A Formal Proof That $\zeta(-1) = -\frac{1}{12}$

Agentic NL→Lean 4 Pipeline

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Abstract

The Riemann zeta function, analytically continued to the entire complex plane except $s = 1$, takes the value $-\frac{1}{12}$ at $s = -1$. This identity underlies the notorious "sum" $1 + 2 + 3 + \cdots = -\frac{1}{12}$ encountered in string theory and regularization. We formally verified the equality $\zeta(-1) = -\frac{1}{12}$ in Lean 4 using *Mathlib*. The proof reduces the evaluation to the Bernoulli number $B_2 = \frac{1}{6}$ via Mathlib's formula for ζ at negative integers.

1 Introduction

Euler discovered that the analytic continuation of $\sum_{n\geq 1} n^{-s}$ assigns the value $-\frac{1}{12}$ to $s = -1$. This value recurs in physics: bosonic string theory uses it to fix the critical dimension, and Casimir-effect computations invoke it to regularize divergent sums. Mathematically, $\zeta(-1) = -\frac{1}{12}$ follows from the functional equation together with the identity

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \quad (n \geq 0),$$

where B_k denotes the k-th *Bernoulli number*. We report a machine-checked proof of this identity in Lean 4.

Theorem 1. $\zeta(-1) = -\frac{1}{12}$.

2 Formal Statement

```
theorem riemannZeta_at_neg_one : riemannZeta (-1) = -1/12
```

3 Natural Language Proof

Let $n = 1$. The standard formula for the Riemann zeta function at negative integers states

$$\zeta(-n) = -\frac{B_{n+1}}{n+1},$$

so in particular

$$\zeta(-1) = -\frac{B_2}{2}.$$

We compute B_2. By the relation between the two conventions of Bernoulli numbers, $B_k = B'_k$ whenever $k \neq 1$, and the classical identity $B'_2 = \frac{1}{6}$ gives $B_2 = \frac{1}{6}$. Substituting,

$$\zeta(-1) = -\frac{1/6}{2} = -\frac{1}{12}.$$

Therefore $\zeta(-1) = -\frac{1}{12}$. □

4 Formal Lean 4 Proof

The proof invokes `riemannZeta_neg_nat_eq_bernoulli`, converts between Bernoulli conventions via `bernoulli_eq_bernoulli'_of_ne_one`, applies `bernoulli'_two`, and closes the arithmetic with `push_cast` and `ring`.

```
import Mathlib

theorem riemannZeta_at_neg_one : riemannZeta (-1) = -1/12 := by
  have h := riemannZeta_neg_nat_eq_bernoulli 1
  have hb : bernoulli 2 = 1/6 := by
    rw [bernoulli_eq_bernoulli'_of_ne_one (by norm_num)]
    exact bernoulli'_two
  simp only [Nat.cast_one, pow_one, Nat.reduceAdd] at h
  rw [h, hb]
  push_cast
  ring
```

5 Conclusion

We reduced $\zeta(-1) = -\frac{1}{12}$ to the Bernoulli-number identity $B_2 = \frac{1}{6}$ and verified both in Lean 4 with Mathlib. The Lean kernel has checked every step, leaving no gap between the informal textbook argument and its formal counterpart.