A Formal Proof That No Rational Number Squares to Two

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Abstract

We state and formally verify the classical fact that no ratio of integers a/b with $b \neq 0$ satisfies $(a/b)^2 = 2$. The proof is machine-checked in Lean 4 against Mathlib. The argument reduces the claim to the irrationality of $\sqrt{2}$, a result already present in Mathlib as `irrational_sqrt_two`. The key takeaway is a compact, auditable certificate for one of the oldest theorems in mathematics.

1 Introduction

The irrationality of $\sqrt{2}$ is the archetypal impossibility result in number theory, traditionally attributed to the Pythagorean school. We formalize a direct corollary: the equation $(a/b)^2 = 2$ has no integer solutions with $b \neq 0$. This statement is the *rational* form of the irrationality of $\sqrt{2}$, phrased without invoking real numbers explicitly. We prove:

Theorem 1. *There do not exist integers a, b with $b \neq 0$ such that $\left(\frac{a}{b}\right)^2 = 2$.*

2 Formal Statement

```
theorem no_rat_sqrt_two : ¬  a b :  , b  0  ((a : ) / (b : ))
    ^2 = 2
```

3 Natural Language Proof

Suppose, for contradiction, that integers a and b exist with $b \neq 0$ and $(a/b)^2 = 2$ in $\mathbb{Q}$. Set $q := a/b \in \mathbb{Q}$. Viewing q as a real number, the hypothesis gives $q^2 = 2$ in $\mathbb{R}$. Taking square roots and using $\sqrt{x^2} = |x|$ for real x, we obtain

$$\sqrt{2} = |q|.$$

Since $q \in \mathbb{Q}$, its absolute value $|q|$ is also rational. Therefore $\sqrt{2}$ equals a rational number, contradicting the *irrationality of* $\sqrt{2}$. Hence no such a, b exist. □

4 Formal Lean 4 Proof

```
import Mathlib

open Real

theorem no_rat_sqrt_two : ¬  a b :  , b  0  ((a :  ) / (b :  ))
    ^2 = 2 := by
  rintro  a, b, hb,  h
  have hirr : Irrational (Real.sqrt 2) := irrational_sqrt_two
  set q :   := (a :  ) / (b :  ) with hq_def
  have hqR : ((q :  ))^2 = 2 := by exact_mod_cast h
  have heq : Real.sqrt 2 = |(q :  )| := by
    rw ←[ hqR, Real.sqrt_sq_eq_abs]
  apply hirr
  rw [heq]
  refine  |q|, ? _
  push_cast
  rfl
```

The proof invokes Mathlib's `irrational_sqrt_two` and `Real.sqrt_sq_eq_abs`, together with the tactics `rintro`, `exact_mod_cast`, `push_cast`, and `rfl` to discharge the casts between $\mathbb{Q}$ and $\mathbb{R}$.

5 Conclusion

We reduced the nonexistence of a rational square root of 2 to Mathlib's packaged irrationality of $\sqrt{2}$ and discharged the reduction through a short chain of casts. The result is machine-verified in Lean 4, yielding a certificate that can be rechecked independently by any Lean user.