A Formal Proof That the Sum of the First n Odd Naturals Equals n^2

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Abstract

We formally verify the classical identity that the sum of the first n odd natural numbers equals n^2. The statement $\sum_{k=0}^{n-1}(2k+1) = n^2$ was mechanically checked in *Lean 4* using the *Mathlib* library. The proof proceeds by induction on n, with the inductive step closed by ring normalization. The result is a machine-verified certificate of one of the oldest identities in elementary number theory.

1 Introduction

The identity $1 + 3 + 5 + \cdots + (2n-1) = n^2$ dates at least to the Pythagoreans, who visualized it as a decomposition of an $n \times n$ square into nested gnomons. It is among the first non-trivial identities a student encounters and a standard proving ground for mathematical induction. We present a formal verification of this identity in Lean 4.

Theorem 1. *For every* $n \in \mathbb{N}$,

$$\sum_{k=0}^{n-1}(2k+1) = n^2.$$

2 Formal Statement

The theorem is stated in Lean 4 as follows.

```
theorem sum_first_n_odds (n : N) : \sum k \in Finset.range n,
    (2 * k + 1) = n ^ 2
```

3 Natural Language Proof

Proof. We proceed by induction on n.

Base case. For $n = 0$, the sum is empty, so $\sum_{k=0}^{-1}(2k+1) = 0 = 0^2$.

Inductive step. Assume the identity holds for some $n \in \mathbb{N}$, i.e. $\sum_{k=0}^{n-1}(2k + 1) = n^2$. We show it holds for $n + 1$. Splitting off the last term,

$$\sum_{k=0}^{n}(2k+1) = \sum_{k=0}^{n-1}(2k+1) + (2n+1).$$

By the inductive hypothesis, the right-hand side equals

$$n^2 + (2n+1) = (n+1)^2.$$

Therefore $\sum_{k=0}^{n}(2k+1) = (n+1)^2$, completing the induction. □

4 Formal Lean 4 Proof

The proof uses induction on n, closes the base case with `simp`, and closes the inductive step by rewriting with `Finset.sum_range_succ` and the inductive hypothesis, followed by `ring`.

```
import Mathlib

open Finset

theorem sum_first_n_odds (n : N) : \sum k \in Finset.range n,
    (2 * k + 1) = n ^ 2 := by
  induction n with
  | zero => simp
  | succ n ih =>
    rw [Finset.sum_range_succ, ih]
    ring
```

5 Conclusion

We verified the Pythagorean identity $\sum_{k=0}^{n-1}(2k + 1) = n^2$ in Lean 4 against Mathlib. The proof is short, the certificate is machine-checked, and the argument mirrors the textbook induction.