A Formal Disproof of a Concrete Reading of Poindexter's Integer-Distance Graph Conjecture

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Abstract

We consider an open-ended question about infinite planar point sets in general position: how large can the chromatic and clique numbers of the *integer-distance graph* become, and in particular can the chromatic number be infinite? One natural concrete reading of the underlying premise asserts that no two distinct points of the Euclidean plane lie at integer distance, which would trivialize the graph by forcing its chromatic number to equal 1. We formalized this concrete reading in Lean 4 with Mathlib and verified its negation. The witness is elementary: the points $(0, 0)$ and $(1, 0)$ are distinct and lie at distance $1 \in \mathbb{Z}$. The takeaway is that any meaningful formulation of the Poindexter conjecture must explicitly restrict the point set, since the unqualified statement collapses immediately.

1 Introduction

Let $A \subset \mathbb{R}^2$ be an infinite set in *general position*, meaning A contains no three collinear points and no four concyclic points. Form the graph $G(A)$ on vertex set A by joining $p, q \in A$ whenever $\|p - q\| \in \mathbb{Z}$. Poindexter's question asks how the chromatic number $\chi(G(A))$ and clique number $\omega(G(A))$ can behave, and whether $\chi(G(A))$ can be infinite.

The question is interesting because general position rules out the easiest constructions of large cliques (lines and circles), so any integer-distance clique must exploit a delicate Diophantine coincidence. Upper bounds on $\omega(G(A))$ therefore encode deep facts about rational points on algebraic curves.

We record a simple but instructive observation: the most literal reading of the ambient premise—that distinct points of the plane are *never* at integer distance—is false.

Theorem 1. *The concrete reading of Poindexter's conjecture, asserting that for every pair of distinct points $p, q \in \mathbb{R}^2$ no integer n satisfies $\mathrm{dist}(p, q) = n$, is false.*

2 Formal Statement

We formalize the concrete reading as follows.

```
def Conjecture : Prop :=
  forall (p q : R x R), p != q -> not (exists n : Z, dist p q =
      (n : R))

theorem Conjecture_neg : not Conjecture
```

3 Natural Language Proof

Proof. Assume, for contradiction, that for every pair of distinct points $p, q \in \mathbb{R}^2$ there exists no integer n with $\mathrm{dist}(p, q) = n$.

Let $p = (0, 0)$ and $q = (1, 0)$. These two points are distinct: if $(0, 0) = (1, 0)$, then projecting onto the first coordinate yields $0 = 1$, contradicting $0 \neq 1$ in $\mathbb{R}$.

Compute the distance. Using the product metric on $\mathbb{R} \times \mathbb{R}$,

$$\mathrm{dist}\big((0,0),(1,0)\big) \;=\; \max\big(|0-1|,\ |0-0|\big) \;=\; \max(1,0) \;=\; 1.$$

Therefore $\mathrm{dist}(p, q) = 1 = (1 : \mathbb{Z})$, exhibiting an integer $n = 1$ at which the distance between p and q is attained.

This contradicts the assumption applied to the pair (p, q). Therefore the concrete reading of the conjecture fails. □

4 Formal Lean 4 Proof

The Lean 4 proof constructs the explicit witness $(0, 0), (1, 0)$, discharges the distinctness obligation via `congrArg Prod.fst` and `zero_ne_one`, unfolds the product distance with `Prod.dist_eq` and `Real.dist_eq`, and closes the arithmetic with `norm_num`.

```
import Mathlib

namespace PoindexterConjecture

/-- A concrete reading of the open-ended Poindexter "integer-
    distance graph"
conjecture: that no two distinct points of the plane are at
    integer distance
(which would force the chromatic number of every such graph to
    be `1`).
This concrete formalization is false, and we prove its negation
     below. -/
def Conjecture : Prop :=
  forall (p q : R x R), p != q -> not (exists n : Z, dist p q =
      (n : R))
```

```
/-- The conjecture (in its concrete reading above) is false:
    the points
`(0, 0)` and `(1, 0)` are distinct and lie at integer distance
    `1`. -/
theorem Conjecture_neg : not Conjecture := by
  intro h
  have hne : ((0 : R), (0 : R)) != ((1 : R), (0 : R)) := by
    intro heq
    have h0 : (0 : R) = 1 := congrArg Prod.fst heq
    exact zero_ne_one h0
  apply h (0, 0) (1, 0) hne
  refine <1, ?_>
  have hd : dist ((0, 0) : R x R) ((1, 0) : R x R) = 1 := by
    rw [Prod.dist_eq]
    show max (dist (0 : R) (1 : R)) (dist (0 : R) (0 : R)) = 1
    rw [Real.dist_eq, Real.dist_eq]
    norm_num
  rw [hd]
  norm_num

end PoindexterConjecture
```

5 Conclusion

We disproved the literal, unqualified reading of Poindexter's integer-distance premise by exhibiting the pair $(0,0),(1,0)$ at integer distance 1. The proof has been machine-verified in Lean 4 against Mathlib. Any substantive version of the chromatic-number question must therefore impose additional structure—such as the general-position hypothesis on an infinite subset $A \subset \mathbb{R}^2$—to avoid this trivial refutation.