A Formal Refutation of "Any Number Divided by Itself Is a Whole Number"

Agentic NL→Lean 4 Pipeline

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Abstract

The folklore claim "any number divided by itself is a whole number" appears self-evident, since $a/a = 1$ whenever $a \neq 0$. We show the claim fails over the real numbers once we include $a = 0$, because Lean's convention fixes $0/0 = 0$, which is not a *positive* whole number. We formalized the refutation in *Lean 4* with *Mathlib*, producing a machine-checked counterexample. The takeaway: even trivial-sounding arithmetic lemmas require careful attention to edge cases and the ambient convention for division by zero.

1 Introduction

Elementary intuition suggests that dividing any number by itself yields 1, a paradigmatic whole number. Formalizing this intuition forces a choice: what does $0/0$ mean? Lean's real-number library, following standard Mathlib convention, defines $0/0 = 0$. Under the natural reading "whole number" = *positive* natural number (so that the value 1 qualifies), the universal statement becomes false at $a = 0$. We prove the negation:

$$\neg\left(\forall a \in \mathbb{R},\ \exists n \in \mathbb{N},\ 0 < n\ \wedge\ a/a = n\right).$$

2 Formal Statement

```
theorem any_number_divided_by_itself_is_whole_neg :
    (forall a : R, exists n : N, 0 < n /\ a / a = (n : R))
```

3 Natural Language Proof

Proof. Suppose, for contradiction, that for every real a there exists a positive natural number n with $a/a = n$. Apply the hypothesis at $a = 0$: there exists $n \in \mathbb{N}$ such that $0 < n$ and $0/0 = n$. By the convention $0/0 = 0$ in the real numbers (as fixed by the Mathlib definition of division), we have $n = 0$. Substituting into $0 < n$ yields $0 < 0$, a contradiction. Therefore the universal statement is false. □ □

4 Formal Lean 4 Proof

The proof uses `simp` to evaluate $0/0 = 0$, `exact_mod_cast` to descend from $\mathbb{R}$ to $\mathbb{N}$, and `Nat.lt_irrefl` to close the contradiction.

```
import Mathlib

open scoped Classical

/--
Formalization note: interpreting "whole number" as a *positive*
     natural number
(which matches the intuitive expectation that a/a = 1, a whole
    number), the
conjecture "any number divided by itself is a whole number" is
    FALSE in Lean's
real arithmetic, because the counterexample a = 0 gives 0/0 =
    0, which is not
a positive whole number.
-/
theorem any_number_divided_by_itself_is_whole_neg :
    (forall a : R, exists n : N, 0 < n /\ a / a = (n : R)) :=
        by
  intro h
  obtain <n, hn_pos, hdiv> := h 0
  have h0 : (0 : R) / 0 = 0 := by simp
  rw [h0] at hdiv
  have hn_real : (n : R) = 0 := hdiv.symm
  have hn_zero : n = 0 := by exact_mod_cast hn_real
  exact (Nat.lt_irrefl 0) (hn_zero > hn_pos)
```

5 Conclusion

The conjecture "any number divided by itself is a whole number" is false over $\mathbb{R}$ under the standard Mathlib convention $0/0 = 0$, with $a = 0$ serving as a counterexample. Lean 4 and Mathlib verified the refutation, yielding a machine-checked guarantee that no positive natural number equals $0/0$. The episode illustrates how formalization surfaces conventions that informal mathematics leaves implicit.