Empirical Investigation of an Open Conjecture: KLS Conjecture (Kannan–Lovász–Simonovits)

Agentic NL→Lean 4 Pipeline
Job #14

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Abstract

This report documents the empirical investigation of an open mathematical conjecture that could not be formally proved or disproved in Lean 4 with Mathlib. Numerical experiments were conducted to gather evidence for or against the conjecture. The empirical verdict is: **Empirically Supported**. The conjecture remains formally open.

1 Conjecture Statement

Conjecture 1.

```
KLS Conjecture (--KannanLovászSimonovits)

For each integer


1
n1, let

 be a Borel probability measure on


R
n
 that is absolutely continuous with respect to Lebesgue measure, with
    density

: →




[
0
,∞

)
f:R
n→∞
[0,) of the form
```

(

)
= -

(

)
f(x)=e-
V(x)
, where

: →

(-∞

,∞

]
V:R
n→-∞∞
(,] is convex; equivalently,

is log-concave. Assume

is isotropic, meaning

(

)
=
0
and

(

)
=

,

R
n

xd(x)=0 and
R
n

xx

d(x)=I
n

,

where

I
n

is the

×

n×n identity matrix.

For a measurable set

AR
n
, define its

-enlargement by

=
{

:
dist

(

,

)

}
A

={ xR
n
:dist(x,A) } and its Minkowski boundary measure (with respect to

) by

+
(

)
=
liminf ↓

0

(

)−

(

)

.

+
(A)= ↓
0
liminf

(A

)- (A)

.

Define the Cheeger (isoperimetric) constant of

as

=
inf

measurable
,

0
<

(

)
<
1

+
(

)
min

{

(

)
,
1-

(

)
}
.

```
=
A measurable, 0 <(A)<1
inf


min {(A)- ,1(A)}

+
(A)



.
Unsolved Problem

(--KannanLovászSimonovits conjecture; Kannan et al. (1995)) determine whether
     there exists a universal constant

>
0
c>0 (independent of

n and of

) such that for every dimension

n and every isotropic log-concave probability measure

 on


R
n
,






.





c.

Equivalently, if


C
n


 is the smallest number such that every isotropic log-concave
```

```
 on


R
n
 satisfies


+
(

)

1


min

{

(

)
,
1-


(

)
}
for all measurable





,

+
(A)
C
n


1


min {(A)- ,1(A)} for all measurable AR
n
,

the conjecture is that
sup
```

1

<∞

sup
n1

C
n∞

< (that is,

=

(
1
)
C
n

=O(1) as →∞→∞

n). Eldan (2013) introduced the stochastic localization approach. The best known general bound is currently

=

(
log

)
C
n

=O(
logn

), due to Klartag (2023).

Solution Claims

```
Accepted clai
... [truncated]
```

2 Status

Formal Status: OPEN — no Lean 4 proof or disproof was found.

Empirical Verdict: Empirically Supported

The pipeline attempted formal verification in Lean 4 with Mathlib but was unable to produce a compiling proof or disproof. Empirical testing was then conducted to gather numerical evidence.

3 Basic Empirical Testing

The following output was produced by the basic numerical experiment:

```
=== EXPERIMENT PLAN ===

Goal: Empirically test the Kannan-Lovász-Simonovits (KLS) conjecture:
  inf_{n>=1} inf_{ isotropic log-concave on R^n}  _  >  0,
where  _ is the Cheeger (Minkowski-boundary-over-mass) constant.

Since we cannot enumerate ALL isotropic log-concave measures, we test a
diverse family of canonical examples across a broad range of dimensions:
  (a) Standard Gaussian                         --  known = √(2/)
  (b) Uniform on cube √√[-3,3]^n                -- product, log-concave
  (c) Uniform on Euclidean ball  (radius √(n+2)) -- rotationally symmetric
  (d) Product Laplace (scale √1/2)              -- heavy-tail log-concave
  (e) Uniform on  1 ball                        -- non-product, convex body
Dimensions tested: n  {2, 5, 10, 20, 50, 100, 200}; plus stress test n=500.

Methodology (multiple complementary angles):
  1) Half-space Cheeger estimate.  For each (, n) we draw N=3000 i.i.d.
     points, whiten them (subtract mean, multiply by Cov^{-1/2}) to enforce
     isotropy EXACTLY on the empirical measure, then for K=30 random unit
     directions u estimate
         R_u = inf_t  f_u(t) / min(F_u(t), 1 - F_u(t))
     via KDE + empirical CDF at quantiles q  [0.03, 0.97].
     Set ^_HS(n ,) = min_u R_u (an upper bound on the true  _).
  2) Counterexample hunt by direction.  For the cube (believed extremal
     candidate) we evaluate axis and diagonal directions explicitly to
     see which directions try to "break" KLS.
  3) Non-half-space cuts.  Coordinate slabs {|x_1 |t} and Euclidean
     balls {|x |r} are evaluated on Gaussian(n=50) to confirm that
     half-spaces are tighter (smaller ratio) than these alternatives.
  4) Asymptotic check.  We plot ^_HS(n) and also ^_HS(n)√·(log n).
     KLS predicts the former flat; Klartag (2023) upper bound is
     C_n=O√(log n), i.e. ^√·log n  const.  A rapid decay of ^ toward
     0 would refute KLS.
  5) Statistical trend test.  Linear regression of ^_HS on log(n) for
     each family; negative slopes of big magnitude would be a warning.
```

```
Verdict rule:
  - SUPPORTED if min over all (,n) tested of ^_HS  0.30 AND the
    regression slopes vs log(n) are NOT sharply negative.
  - REFUTED if we find ^ collapsing toward 0 in a dimension-growing way.
  - INCONCLUSIVE otherwise.

=== RUNNING: half-space Cheeger estimates ===
    n |         Gaussian |     Uniform cube |     Uniform ball |     Laplace
        prod |  Uniform 1-ball
------------------------------------------------------------------------------------------------

    2 |           0.7935 |           0.5736 |           0.6259 |
       0.9676 |           0.5839
    5 |           0.7596 |           0.6052 |           0.6824 |
       0.8484 |           0.6955
   10 |           0.7465 |           0.6961 |           0.7176 |
      0.7931 |           0.6960
   20 |           0.7500 |           0.7470 |           0.7405 |
      0.7850 |           0.7259
   50 |           0.7472 |           0.7564 |           0.7382 |
      0.7857 |           0.7265
  100 |           0.7445 |           0.7553 |           0.7369 |
     0.7477 |           0.7404
  200 |           0.7578 |           0.7343 |           0.7453 |
     0.7685 |           0.7565

Total directional trials (half-space cuts): 1050
Theoretical Gaussian Cheeger: √(2/) = 0.7979

=== BENCHMARK: non-half-space cuts (Gaussian, n=50) ===
  Coordinate slab {|x1|<=t}   min ratio = 1.2821
  Euclidean ball  {|x|<=r}   min ratio = 1.1598
  Half-space est at n=50                 = 0.7472
  (Half-space < slab/ball -> half-spaces are the tighter test sets,
     consistent with KLS expectations.)

=== COUNTEREXAMPLE HUNT: extremal directions in the cube (n=100) ===
  axis direction (worst?): ^  = 0.5807  (theory: uniform√√[-3,3] => √1/3
     0.5774)
  diagonal direction (CLT): ^ = 0.8032  (theory: ~Gaussian => √(2/)   0.7979)
  random direction: ^         = 0.7675

=== HIGH-DIM STRESS TEST: n = 500 ===
          Gaussian, n=500:
... [truncated]
```

4 Experiment Code (Basic)

```
import numpy as np
import matplotlib
```

```
matplotlib.use("Agg")
import matplotlib.pyplot as plt
from scipy import stats
import time

np.random.seed(42)
rng = np.random.default_rng(42)

start = time.time()

print("===␣EXPERIMENT␣PLAN␣===")
print("""
Goal: Empirically test the Kannan-Lovász-Simonovits (KLS) conjecture:
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""")

# ----------------------- Samplers ------------------------
```

```
def sample_gauss(N_, n):
    return rng.standard_normal((N_, n))

def sample_cube(N_, n):
    return rng.uniform(-np.sqrt(3.0), np.sqrt(3.0), (N_, n))

def sample_uniform_ball(N_, n):
    R = np.sqrt(n + 2.0)
    Y = rng.standard_normal((N_, n))
    Y /= np.linalg.norm(Y, axis=1, keepdims=True)
    u = rng.random(N_)
    r = R * u**(1.0/n)
    return r[:, None] * Y

def sample_laplace(N_, n):
    return rng.laplace(0.0, 1.0/np.sqrt(2.0), (N_, n))

def sample_uniform_l1_ball(N_, n):
    R = np.sqrt((n + 1.0) * (n + 2.0) / 2.0)
    E = rng.exponential(1.0, (N_, n + 1))
    S = E.sum(axis=1, keepdims=True)
    T = R * E[:, :n] / S
    signs = rng.choice(np.array([-1.0, 1.0]), size=(N_, n))
    return signs * T

distros = {
    "Gaussian":          sample_gauss,
    "Uniform␣cube":      sample_cube,
    "Uniform␣ball":      sample_uniform_ball,
    "Laplace␣prod":      sample_laplace,
    "Uniform␣1-ball":  sample_uniform_l1_ball,
}

# ----------------------- Helpers -------------------------

def isotropize(X):
    Xc = X - X.mean(axis=0)
    if X.shape[1] == 1:
        s = Xc.std(ddof=0)
        return Xc / max(s, 1e-12)
    C = np.cov(Xc, rowvar=False)
    vals, vecs = np.linalg.eigh(C)
    vals = np.maximum(vals, 1e-12)
    W = (vecs * vals**(-0.5)) @ vecs.T
    return Xc @ W

def halfspace_cheeger_estimate(X, n_dirs=30, q_lo=0.03, q_hi=0.97, n_q=60):
    N_, n = X.shape
    U = rng.standard_normal((n_dirs, n))
    U /= np.linalg.norm(U, axis=1, keepdims=True)
    Z = X @ U.T
    qs = np.linspace(q_lo, q_hi, n_q)
    ratios = np.zeros(n_dirs)
```

```
    for k in range(n_dirs):
        z = Z[:, k]
        kde = stats.gaussian_kde(z, bw_method="silverman")
        ts = np.quantile(z, qs)
        fs = kde(ts)
        rs = fs / np.minimum(qs, 1.0 - qs)
        ratios[k] = rs.min()
    return ratios.min(), ratios

# ---------------------- Main sweep ----------------------

dims = [2, 5, 10, 20, 50, 100, 200]
N_SAMPLES = 3000
N_DIRS = 30

results = {name: {"psi_min": [], "psi_mean": [], "psi_std": []}
           for name in distros}
total_trials = 0

print("=== RUNNING: half-space Cheeger estimates ===")
header = f"{'n':>5} | " + " | ".join(f"{name:>16}" for name in distros)
print(header)
print("-" * len(header))
for n in dims:
    row = [f"{n:>5}"]
    for name, sampler in distros.items():
        X = sampler(N_SAMPLES, n)
        X = isotropize(X)
        psi_hat, per_dir = halfspace_
# ... [truncated]
```

5 Conclusion

The conjecture remains formally open. Numerical experiments **support** the conjecture — no counterexamples were found across all tested parameter ranges. Further investigation (both formal and empirical) is warranted.