A Formal Proof of the Pythagorean Theorem in Lean 4

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Abstract

We formalize the classical Pythagorean theorem: in any Euclidean right triangle with vertices A, B, C and a right angle at C, the squares of the legs AC and BC sum to the square of the hypotenuse AB. The result is stated in the generality of a real inner product space acting on a metric space via a *normed additive torsor*, so it applies to every Euclidean affine space. We verify the proof in *Lean 4* using *Mathlib*. The proof reduces to a single equivalence already present in Mathlib's Euclidean geometry library, which converts the right-angle hypothesis into the required distance identity.

1 Introduction

The Pythagorean theorem is the foundational metric identity of Euclidean geometry. Stated in modern language: if A, B, C are points in a Euclidean affine space and the angle $\angle ACB = \pi/2$, then

$$\mathrm{dist}(A, C)^2 + \mathrm{dist}(B, C)^2 = \mathrm{dist}(A, B)^2.$$

We record a machine-checked proof of this statement at the level of generality supported by Mathlib: P is a metric space acted on by a real inner product space V through a normed additive torsor structure. This covers $\mathbb{R}^n$ with its standard inner product, as well as every affine Euclidean space modeled on a real Hilbert space.

2 Formal Statement

```
theorem pythagoras {V P : Type*} [NormedAddCommGroup V] [
    InnerProductSpace R V]
    [MetricSpace P] [NormedAddTorsor V P]
    (A B C : P) (h : angle A C B = pi / 2) :
    dist A C ^ 2 + dist B C ^ 2 = dist A B ^ 2
```

3 Natural Language Proof

Theorem 1 (Pythagoras)**.** *Let V be a real inner product space and let P be a Euclidean affine space modeled on V. For points $A, B, C \in P$, if $\angle ACB = \pi/2$, then*

$$\mathrm{dist}(A, C)^2 + \mathrm{dist}(B, C)^2 = \mathrm{dist}(A, B)^2.$$

Proof. Set $u = A - C$ and $v = B - C$, viewed as vectors in V. Then $\mathrm{dist}(A, C) = \|u\|$, $\mathrm{dist}(B, C) = \|v\|$, and $\mathrm{dist}(A, B) = \|u - v\|$. Expanding the squared norm using the inner product gives

$$\|u - v\|^2 = \langle u - v,\, u - v\rangle = \|u\|^2 - 2\langle u, v\rangle + \|v\|^2.$$

By hypothesis, $\angle ACB = \pi/2$, so the vectors u and v are orthogonal; equivalently, $\langle u, v\rangle = 0$. Substituting yields

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2,$$

i.e.

$$\mathrm{dist}(A, B)^2 = \mathrm{dist}(A, C)^2 + \mathrm{dist}(B, C)^2.$$

Rearranging gives the claimed identity. $\square$ $\square$

4 Formal Lean 4 Proof

The proof invokes Mathlib's equivalence `EuclideanGeometry.dist_sq_eq_dist_sq_add_dist_sq_iff_angle_` which converts the right-angle hypothesis into the squared-distance identity, and closes the remaining linear arithmetic with `linarith`.

```
import Mathlib

open EuclideanGeometry Real

/--
Pythagoras' theorem: for a Euclidean right triangle with
    vertices `A`, `B`, `C`
and right angle at `C`, the squares of the legs `AC` and `BC`
    sum to the square
of the hypotenuse `AB`.
-/
theorem pythagoras {V P : Type*} [NormedAddCommGroup V] [
    InnerProductSpace ℝ V]
    [MetricSpace P] [NormedAddTorsor V P]
    (A B C : P) (h : angle A C B = pi / 2) :
    dist A C ^ 2 + dist B C ^ 2 = dist A B ^ 2 := by
  have key :=
    (EuclideanGeometry.
        dist_sq_eq_dist_sq_add_dist_sq_iff_angle_eq_pi_div_two
      (p1 := A) (p2 := C) (p3 := B)).mpr h
  linarith
```

5 Conclusion

We verified the Pythagorean theorem in Lean 4 against Mathlib, in the full generality of a real inner product space acting on a metric space as a normed additive torsor. The formalization is short because Mathlib already supplies the precise equivalence between orthogonality of displacement vectors and the squared-distance identity. The result is machine-checked: its correctness rests on Lean 4's kernel and Mathlib's axioms, not on human inspection of the proof script.